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Two approaches to study essentially nonlinear and dispersive properties of the internal structure of materials

A. V. Porubov,[1,*] E. L. Aero,[1,†] and G. A. Maugin[2,‡]

[1]*Institute of Problems in Mechanical Engineering, Bolshoy av. 61, V.O., Saint-Petersburg 199178, Russia*

[2]*Institut Jean le Rond d'Alembert, UMR 7190 CNRS, Université Pierre et Marie Curie (Paris 6), Case 162, Tour 55, Place Jussieu 4, F-75252 Paris Cedex 05, France*

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It is shown that essentially nonlinear models for solids with complex internal structure may be studied using phenomenological and proper structural approaches. It is found that both approaches give rise to the same nonlinear equation for traveling longitudinal macrostrain waves. However, presence of the connection between macro- and microfields in the proper structural model prevents a realization of some important solutions. First, it is obtained that simultaneous existence of compression and tensile waves is impossible in contrast to the phenomenological approach. Then, it is found that correlation between macro- and internal strains gives rise to the crucial influence of the velocity of the macrostrain wave on the existence of either compression or tensile localized strain waves. Also, it is shown that similar profiles of the macrostrain solitary waves may be accompanied by distinct profiles of the microstrain waves. Finally, the dispersion curves for the waves belong to the different branches. This is important for internal structural deviations caused by the dynamical loading due to the localized macrostrain wave propagation and demonstrates a need in the development of the proper structural approaches relative to the phenomenological ones.

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I. INTRODUCTION

The interest in the dynamic behavior of microstructured materials has grown tremendously with the recent manufacture and synthesis of new manifold strongly discrete and layered systems, artificial lattices of nanodots, and other microelements while the classical theories of elasticity and magnetism cannot explain some of the experimental data concerning the properties of such materials with complicated microstructure. This is particularly true for materials of new types such as nanocrystalline alloys, ceramic composites, some biological materials (tissues), multilayered magnetic films and elastic plates, artificial lattices, elastic and optical wave guides with a net of passive and active elements or corrugated internal structure, granular materials, and compounds exhibiting damage under experimental conditions of high speeds of deformation (or high frequencies of vibrations). Similar problems arise for the description of soils and rocks but at the macroscale.

The main difficulties in obtaining the solutions of these problems are related to the existence of intrinsic properties such as additional degrees of freedom, geometric restriction, and geometric characteristics such as size of grains, periodicity of multiatomic lattice or a layered structure, periodicity of active nanoelements, etc. Considerable progress was achieved in the linear description of static states. However, an important problem is to explain how these static states are achieved as time goes on, hence to describe a complex dynamic behavior. Also, most of the parameters are usually obtained using dynamical methods based on the measurements of the velocity of acoustic waves of various polarizations. In particular, these are the higher-order elastic constants of many isotropic and crystalline solids that may be found in textbooks on elasticity. The dynamical modeling of materials with internal structure is not well developed, and we are missing values of the parameters of the existing models.

From the point of view of dynamical processes these parameters characterize nonlinear, dispersive, and dissipative properties of the materials with internal structure. These properties may be recognized by means of the strain waves' behavior. Among nonlinear strain waves of special interest are those that propagate keeping their shape and velocity. One of them is the bell-shaped solitary wave arising as a result of a balance between nonlinearity and dispersion. It is important to find the conditions required for the existence of such a solitary wave in order to know when localization of the strain field is possible. On the other hand, existence of such localized strain waves allows us to estimate the unknown material parameters measuring the wave amplitude and velocity since the relationships between the wave and the material parameters may be found in an explicit form.

To describe strain solitary waves, it is necessary to reveal the sources of nonlinearity and dispersion. The classic elastic theory admits two main kinds of nonlinearity. The first one is the geometrical nonlinearity following from the exact relationship for the tensor of finite strains. An anharmonicity in interatomic interaction gives rise to the so-called physical nonlinearity. Contrary to the geometrical nonlinearity, it is not described by an exact analytical formula but is modeled by implementing some hypothesis about deformations. The most popular of them is to consider truncated power series expansion in strains. This approach is correct for the weakly nonlinear processes as often happens for classic elastic materials [1–6].

*porubov.math@mail.ioffe.ru

†16aero@mail.ru

‡gerard.maugin@upmc.fr

Another source of nonlinearity is caused by the presence of the components with contrasting elastic features including cracks, intergranular contacts, dislocations at the grain boundaries of polycrystals. It was found in Refs. [7–9]. that this nonlinearity is essential, and the contribution of the quadratic and cubic nonlinearities turns out to be of the same order for materials with essentially non linear features. Then the weakly nonlinear models in the form of power series truncation cannot be applied in a strict sense. Nevertheless, they are used already as the exact expressions in the framework of a *phenomenological* approach [7–10].

Another approach in the modeling of essentially nonlinear processes is to apply a proper model for the internal structure of the material. Thus, the rotatory molecular groups were added to the usual one in atomic chain in Refs. [3,11,12], and large rotations were considered. A more complicated internal motion is modeled in Refs. [13,14], where translational internal motion is considered together with rotations. In both models this essential nonlinearity has not been modeled by any power series truncation.

Dispersion in classic elastic bodies is caused by finite transverse sizes of a waveguide. To see how strong this dispersion is, one can note experimental observations of the strain solitary wave of the amplitude of order 10^{-4} and the width 33 mm in a rod of radius $R=5$ mm [5]. Also dispersion appears as a result of an internal complex structure of the material [15–19]. Some estimates have been given for materials with grains and for sandstones [20,21]. According to them the solitary waves may exist in such media with anticipated typical width 0.1–100 m. Finally, one can note phonon dispersion arising in crystals that gives rise to the strain solitary wave of the amplitude of order 10^{-4} and a width of 100 Å observed in experiments [22–24].

Therefore, dispersion may be detected and measured even for media with complex internal structure. This is not true for definitions of the values of the parameters characterizing essential nonlinearity. That is why the simplified modeling with an exploitation of a phenomenological approach is used. However, a natural question arises whether this approach reflects correctly the processes in media with structure.

The main aim of this paper is to compare phenomenological and proper structural modeling of essentially nonlinear processes using the strain solitary wave solutions. Before doing so a review of some known results of the weakly nonlinear modeling is presented in Sec. II to demonstrate the efficiency of the solitary wave solution as a tool for the study of nonlinear dynamical processes. Then this methodology is extended to the essentially nonlinear processes where some models are developed. First, the phenomenological model from Refs. [7–9] is used in Sec. III to derive the governing equations for the strain waves in a medium with essentially nonlinear properties. Then the similar equation is revealed in description of the waves in paramagnetic crystals. Similarly, Sec. III is devoted to obtaining the governing equation for the macrostrain waves based on two proper structural models developed in [3,11,12] and in [13]. The nonlinear and dispersive features of traveling wave solutions of both the phenomenological and structural models are studied and compared in Sec. IV.

II. WEAKLY NONLINEAR MODELING

To illustrate the weakly nonlinear modeling of strain waves, some examples are presented below chosen so as to avoid complicated algebraic manipulations. Most popular are the weakly nonlinear models based on a power series expansion of the energy density in small strains or strain tensor invariants. These series may be truncated since small strains only are considered. This allows us to obtain simpler governing nonlinear equations for strains.

The Murnaghan model [1] may be noted among the models valid for isotropic materials like metals, polymers, etc. In particular, the series truncated at the fourth order term is called the nine-constant Murnaghan model, and its energy density Π reads

$$\Pi = \frac{\lambda+2\mu}{2}I_1^2 - 2\mu I_2 + \frac{l+2m}{3}I_1^3 - 2mI_1I_2 + nI_3 + \nu_1 I_1^4 + \nu_2 I_1^2 I_2 + \nu_3 I_1 I_3 + \nu_4 I_2^2, \quad (1)$$

where $I_k, k=1,2,3$ are the invariants of the Cauchy-Green strain tensor; the fourth-order elastic moduli $(\nu_1,\nu_2,\nu_3,\nu_4)$, as well as the third-order ones, l,m,n, may be of either sign contrary to the positive second-order moduli λ and μ. One can obtain the stress-strain relationship from Eq. (1) in the one-dimensional (1D) case, $P=E^* U_x + C_1 U_x^2 + C_2 U_x^3$, ($U_x$ is the longitudinal strain, E^* is the Young modulus or a combination of the Lamé constants), where the second term describes the quadratic nonlinearity, and $C_1=C_1(l,m,n)$. The last term accounts for the cubic nonlinearity, and its coefficient depends on both the third- and the fourth-order moduli. Usually, only quadratic nonlinearity [the so-called five-constant Murnaghan model with $\nu_i=0$ in Eq. (1)] is used to describe longitudinal strain waves since the relative contribution C_2U_x/C_1 of the last term in the stress-strain relationship is negligibly small for usual elastic materials. Then the governing equation for longitudinal strain waves $v(x,t)=U_x$ is obtained using the Hamilton variational principle. A wave guide strain wave propagation should be considered to involve dispersion, then for longitudinal strain waves in a rod the governing equation is [2–6]

$$v_{tt} - av_{xx} - c_1(v^2)_{xx} + \alpha_3 v_{xxtt} - \alpha_4 v_{xxxx} = 0, \quad (2)$$

where

$$a = \frac{E}{\rho_0}, \quad c_1 = \frac{\beta}{2\rho_0}, \quad \alpha_3 = \frac{\nu(1-\nu)R^2}{2}, \quad \alpha_4 = \frac{\nu E R^2}{2\rho_0},$$

ν is the Poisson ratio, $\beta=3E+2l(1-2\nu)^3+4m(1-2\nu)(1+\nu)^2+6n\nu^2$. Here the dispersion coefficients α_3 and α_4 are calculated using the value of the radius of the rod and the material elastic properties. Equation (2) is often called the double dispersive equation (DDE).

The Murnaghan model may be extended to the crystals [1,3,12,25]. In particular, a 1D problem for the cubic crystal along the direction [100] is described by the energy expression for the displacement $U(x,t)$ of the form

$$\Pi = \frac{1}{2}c_{11}U_x^2 + \frac{1}{6}(3c_{11}+c_{111})U_x^3 + \frac{1}{24}(3c_{11}+6c_{111}+c_{1111})U_x^4, \tag{3}$$

where c_{11}, c_{111}, and c_{1111} are the elastic constants of the second, third, and fourth order. Like in an isotropic case, the fourth-order terms are usually neglected. It allows Hao and Maris [22,23] to derive the Boussinesq equation to describe their experimental observations of strain solitary waves in crystals,

$$\rho_0 v_{tt} - c_{11}v_{xx} - 0.5(3c_{11}+c_{111})(v^2)_{xx} - 2\rho_0 c\gamma v_{xxxx} = 0, \tag{4}$$

where ρ_0 is the density in the underformed state, c is the sound velocity [22]. Contrary to Eq. (2), dispersion now arises from the finite spacing of the atoms, the coefficient γ is calculated using the measurements of the wave front oscillations [26].

The internal structure of material is often taken into account considering the development of the so-called Mindlin model for a weakly nonlinear case. One way is to add the linear gradient term in the classic Murnaghan expansion [15]. This last term gives rise to the dispersion, and the governing equation for a medium is obtained in the form of the DDE (2). The dispersion term coefficients are measured for some materials [20,21].

Despite the DDE being nonintegrable, particular localized traveling wave solutions may be easily obtained by direct integration of it resulting in the first-order ordinary differential equation (ODE)

$$v_\theta^2 = \frac{1}{6(\alpha_3 V^2 - \alpha_4)}[\alpha_0 + \alpha_1 v + 6(V^2-a)v^2 - 4c_1 v^3], \tag{5}$$

where α_0, α_1 are constants of integration, $\theta = x - Vt$. The last equation possesses the well-known solitary wave solution provided that $\alpha_0=0$, $\alpha_1=0$ [5,6]

$$v = \frac{3(V^2-a)}{2c_1}\cosh^{-2}[k(x-Vt)], \tag{6}$$

with k given by

$$k^2 = \frac{V^2-a}{2\nu R^2(a-(1-\nu)V^2)}. \tag{7}$$

Numerical simulation of DDE (2) [5,6] confirms the predictions of the exact solutions. The most important prediction is the dependence of the kind of the localized wave (tensile or compression) on the sign of the nonlinear term coefficient that is defined by the elastic constants of a material (the Murnaghan moduli). In particular, for positive c_1, only tensile exact solitary wave solution exists. In this case only tensile input in numerics splits into a sequence of tensile localized waves each described by Eq. (6) while negative input is dispersed without localization of the waves.

A more complicated governing equation arises when a microfield is introduced together with a macrofield [18,27,28]. Then the Murnaghan model is extended by adding the terms depending on the microfield describing the influence of microstructure. Thus, the expression (with notation used in Refs. [18,27,28]),

$$\Pi = \frac{1}{2}(AU_x^2 + B\psi^2 + C\psi_x^2) + D\psi U_x + \frac{1}{6}NU_x^3, \tag{8}$$

yields the coupled governing equations for the macrodisplacement $U(x,t)$ and microstrain $\psi(x,t)$ in the form

$$\rho U_{tt} - AU_{xx} = NU_x U_{xx} + D\psi_x, \tag{9}$$

$$I\psi_{tt} - C\psi_{xx} = -DU_x - B\psi, \tag{10}$$

where I is the microinertia. Again the nonlinearity is introduced using the truncated series in U_x. Uncoupling may be done in a different manner. Since the truncated expression of the energy is used to obtain Eqs. (9) and (10), these may be decoupled in an asymptotic manner without traveling wave solution assumption. The so-called slaving asymptotic procedure should be applied like in [18] when smallness of the strains is taken into account. Then Eq. (10) yields the expression for ψ,

$$\psi = -\frac{D}{B}U_x + \frac{D}{B^2}(IU_{xtt} - CU_{xxx}),$$

whose substitution to Eq. (9) gives rise to the following DDE for the strain function v:

$$\rho v_{tt} - \left(A - \frac{D^2}{B}\right)v_{xx} = \frac{N}{2}(v^2)_{xx} + \frac{D^2}{B^2}(Iv_{xxtt} - Cv_{xxxx}).$$

Here the dispersion terms arise due to the microstructure contrary to the case of classic elastic bodies (e.g., rods) where dispersion is caused by the boundaries of a waveguide.

Another way was not used in Ref. [18] but linearized version of Eqs. (9) and (10) was transformed in such a way in Ref. [19]. Let us differentiate Eq. (10) with respect to x and use the expression for ψ_x from Eq. (9),

$$\psi_x = \frac{1}{D}(\rho U_{tt} - AU_{xx} - NU_x U_{xx})$$

to eliminate ψ_x in differentiated Eq. (10). Then the governing equation for the strain function $v = U_x$ reads

$$\rho B v_{tt} - (AB - D^2)v_{xx} - \frac{BN}{2}(v^2)_{xx} + \left(I\frac{\partial^2}{\partial t^2} - C\frac{\partial^2}{\partial x^2}\right)\left[\rho v_{tt} - Av_{xx} - \frac{N}{2}(v^2)_{xx}\right] = 0. \tag{11}$$

Usually the last nonlinear term is negligibly small and Eq. (11) generalizes DDE by a more complicated dispersion terms. Then the DDE traveling solitary wave solution (6) holds for Eq. (11),

$$v = \frac{12(\rho BV^2 - AB + D^2)}{BN}\cosh^{-2}[k(x-Vt)], \tag{12}$$

with k given by

$$k^2 = \frac{(\rho BV^2 - AB + D^2)}{4(\rho V^2 - A)(C - IV^2)}. \tag{13}$$

In the general case Eq. (11) may be integrated when traveling wave solutions are considered. Transformation to the phase variable θ gives rise to the ODE for $v(\theta)$ of the form

$$v_\theta^2 = \frac{Bv^2(v^2+pv+q)}{4(C-IV^2)(v+s)^2}, \tag{14}$$

where

$$s = \frac{A-\rho V^2}{N}, \quad p = -\frac{4[2D^2-3B(A-\rho V^2)]}{3BN},$$

$$q = \frac{4(\rho V^2 - A)[D^2+B(\rho V^2-A)]}{BN^2}.$$

One can see that Eq. (14) differs from the ODE reduction of DDE (5), and exact localized wave solution of form (6) does not exist anymore. However, Eq. (14) may be integrated in a standard manner giving localized bell-shaped wave solutions in an implicit form. In particular, such a solution for the case $s>0, B>0$ is given by

$$v = \frac{v_1 - v_2 t^2}{1-t^2}; \quad \ln\left|\frac{1+t}{1-t}\right| + \frac{s}{\sqrt{v_1 v_2}}\ln\left|\frac{\sqrt{v_1}+\sqrt{v_2}t}{\sqrt{v_1}-\sqrt{v_2}t}\right|$$
$$= \frac{2}{N}\sqrt{\frac{C-IV^2}{B}}\theta,$$

where $v_2 > v_1$ are the real positive roots of equation $v^2+pv+q=0$. The solutions of Eq. (14) may be also studied using the phase portraits technique.

A nonlinearity may be involved also at the microlevel. Extra nonlinear terms are added in the expression for energy (8) in a phenomenological way [28]. Additional rotation degree of freedom being taken into account gives rise to the nonlinear terms caused by microstructure [29],

$$\rho U_{tt} - c_1^2 U_{xx} = \frac{1}{2}\frac{\partial}{\partial x}\left(\alpha_1 U_x^2 + \frac{l^2}{a^2}\phi^2\right), \tag{15}$$

$$a^2(\phi_{tt} - c_3^2\phi_{xx}) = -\beta_1\phi - \alpha_3 U_x\phi, \tag{16}$$

where a, l are the parameters of the lattice [29], $\phi(x,t)$ is the angle of rotation. Implementation of the slaving principle is not effective in this case. Instead, the function ϕ is expressed through U using the first equation (this was not considered in Ref. [29]),

$$\phi = \frac{\sqrt{2}a}{\sqrt{\alpha_3}l}[(V^2-c_1^2)U_x - 0.5\alpha_1 U_x^2 + Q_1]^{1/2}.$$

Substituting the last expression into Eq. (16) yields the equation for the strain function $v=U_x$. However, it is rather complicated in the general case. Considering only localized traveling wave solutions depending on the phase variable $\theta = x - Vt$, one obtains the following ODE for $v(\theta)$,

$$v_\theta^2 = \frac{4\alpha_3 v^2(v+2s)(v^2+pv+q)}{3a^2(V^2-c_3^2)(v+s)^2}, \tag{17}$$

where

$$s = \frac{c_1 - V^2}{\alpha_1}, \quad p = \frac{3}{4}\left(s + \frac{\beta_1}{\alpha_3}\right), \quad q = \frac{3\beta_1}{2\alpha_3}.$$

The periodic solution of Eq. (17) may be found using the reference book [30]. Particular cases corresponding to the bell-shaped solitary waves are obtained in an implicit form like the solutions to Eq. (14).

To sum up, the use of the truncated series allows us to obtain the governing equations suitable for an analysis at least in the 1D case. Particular localized wave solutions give the conditions of the strain localizations which are realized in numerical simulations of a more general unsteady process of the strain evolution. The next section addresses the question whether such an analysis is possible when nonlinearity is not weak.

III. MODELS FOR ESSENTIALLY NONLINEAR STRAIN WAVES

In this section we obtain model nonlinear equations for the case when power series truncations become invalid in a strict sense. Below two kinds of modeling of nonlinearity and dispersion of the media which are then essentially nonlinear will be considered.

A. Phenomenological model for seismic waves in rocks and media with microstructure

To obtain the governing equation for seismic essentially nonlinear waves in the 1D case, the stress-strain relationship suggested in [7–9,31] will be used of the form

$$P = E^* U_x + C_1 U_x^2 + C_2 U_x^3, \tag{18}$$

where P is the longitudinal stress, $U(x,t)$ is the longitudinal displacement, E^* is the Young modulus or a combination of the Lamé constants, C_1, C_2 are the parameters either expressed through the linear combinations of the third- and fourth-order elastic moduli in Eq. (1) or measured directly. In the 1D case the Murnaghan model yields

$$E^* = \lambda + 2\mu, \quad C_1 = \frac{1}{2}[3(\lambda+2\mu) + 2(l+2m)],$$

$$C_2 = \frac{1}{2}[\lambda + 2\mu + 4(l+2m) + 8\nu_1].$$

A classic elastic material such as aluminum yields $C_1/E \sim 10$ and $C_2/E \sim 100$ [32] that makes contribution of the third term in Eq. (18) small in comparison with that of the second one for typical strains $U_x \sim 10^{-3}$–10^{-5}. That is why cubic nonlinear terms are neglected for modeling of longitudinal strain waves.

This is not true for some rocks, sandstones, limestones and granites, which is shown in Table I. The third-order constants exceed the Lamé coefficients by 2 or even 4 orders. This illustrates the phenomenon of abnormal nonlinearity [8,9], when even strains of order 10^{-4} should be considered as nonlinear. At the same time the materials from Table I

TABLE I. Lamé's and Murnaghan's moduli in 10^{10} N/m^2 for some rocks (after [10]) and contribution of the quadratic nonlinearity.

Material	λ	μ	l	m	C_1/E^*
Cemented glass beads	0.6	0.55	−71.6	−67.8	−120.4
Sandstone Massilon SS	0.19	0.63	−7900	−1443.5	−2500
Limestone 1083	2.3	2.1	−830.2	−1130	−470
Granite Westerly	2.2	2.4	−2137.7	−2726.3	−1080

have the Lamé coefficients of the same order as of the classic elastic materials.

The direct measurement of the coefficients in Eq. (18) for a medium with tuff grains [8] yields $C_1/E^* \sim 130$, $C_2/E^* \sim 10^4$, that gives rise to almost equal contributions of quadratic and cubic nonlinearities for the observed strains of order 10^{-4}. Similar estimates follow using the data for a loam soil [8] and for a soft fault core [33] where $C_1/E^* \sim 10^3$, $C_2/E^* \sim 10^7$ [8]. The relationship (18) was used in [31] to model a medium with cracks; the estimates obtained there are $C_1/E^* \sim 10^2$, $C_2/E^* \sim 10^8$.

According to Refs. [7–9,31] the source of abnormal nonlinearity is the presence of the components with contrasting compressibility in the materials like shown in Table I. Usual weak nonlinearity is caused by the anharmonicity in interatomic interactions. Among the soft inclusions one can note cracks, intergrain contacts, pores, and accumulation of dislocations at the grain boundaries of polycrystals [7].

The data presented before demonstrates uselessness of the truncated power series expressions of strains in the expressions for the energy and the strain used in the weakly nonlinear theory. Nevertheless, the *phenomenological* approach of the essential nonlinear processes is based on the formal use of Eq. (1) or Eq. (18) but as exact representations of the energy or the strain [7–9,31].

Recently, a theory has been developed that takes into account the major role of hysteretic nonlinearity [34] that gives rise to a more complicated expression than Eq. (18). However, analysis in Refs. [7–9,31] as well as the modeling in Ref. [33] allows us to use Eq. (18) nevertheless especially because the values of C_1 and C_2 found in these papers yield the model based on Eq. (18) be suitable for real materials.

Dispersion in seismic media is brought about for various reasons but essentially in the same manner as in the weakly nonlinear theory. The higher-order derivative terms may be added to Eq. (1) or Eq. (18) to describe coupled stresses arising due to an internal structure, e.g., rotation of the particles [21]. In layered seismic structures dispersion is caused by the finite size of a layer [21,33].

Now we apply the algorithm of the weakly nonlinear theory to obtain the governing equation. The Hamilton variational principle with the Murnaghan expression for the energy density (1) in the general 3D statement when the boundary conditions are important, in particular, when a layer or another wave guide is considered. The 1D statement for a medium allows us to employ also Eq. (18) for the stresses by substituting it into the 1D equation of motion. In the last case dispersion is added phenomenologically like it is done in the weakly nonlinear case in Refs. [22,23]. In any case the resulting governing equation for $v(x,t)=U_x$ is obtained in the form

$$v_{tt} - av_{xx} - c_1(v^2)_{xx} - c_2(v^3)_{xx} + \alpha_3 v_{xxtt} - \alpha_4 v_{xxxx} = 0, \tag{19}$$

where $a=E/\rho$. Relationships for nonlinear term coefficients are expressed either through the Murnaghan moduli of the third and the fourth order and they depend upon the boundary conditions. In the 1D statement they are given by the coefficients in Eq. (18), $c_1=2C_1/\rho$ and $c_2=3C_2/\rho$. One can see that this equation modifies the DDE Eq. (2) by only one cubic nonlinear term.

The Eq. (19) is nonintegrable but traveling localized wave solutions can be found by introducing the phase variable $\theta = x - Vt$ and transforming Eq. (19) to the following ODE:

$$v_\theta^2 = \frac{1}{6(\alpha_3 V^2 - \alpha_4)}[\alpha_0 + \alpha_1 v + 6(V^2 - a)v^2 - 4c_1 v^3 - 3c_2 v^4], \tag{20}$$

where α_0, α_1 are constants of integration. This equation is similar to the ODE reduction of the Gardner equation that accounts for internal shear waves in two-layer fluids [35]. In fluids the equality of the contribution of quadratic and cubic nonlinear terms happens for very special ratio between the width of the layers and their densities (this is a weakly nonlinear problem) [35].

Using above-mentioned analysis, one can suggest the following generalization of the weakly nonlinear model [Eqs. (9) and (10)],

$$\rho U_{tt} - AU_{xx} = NU_x U_{xx} + MU_x^2 U_{xx} + D\psi_x, \tag{21}$$

$$I\psi_{tt} - C\psi_{xx} = -DU_x - B\psi. \tag{22}$$

Like in the weakly nonlinear case, the governing equation for the macrostrain is obtained in the form

$$\rho B v_{tt} - (AB - D^2)v_{xx} - \frac{BN}{2}(v^2)_{xx} - \frac{BM}{3}(v^3)_{xx} + \left(I\frac{\partial^2}{\partial t^2} - C\frac{\partial^2}{\partial x^2}\right)\left[\rho v_{tt} - Av_{xx} - \frac{N}{2}(v^2)_{xx} - \frac{M}{6}(v^3)_{xx}\right] = 0. \tag{23}$$

Usually the last nonlinear terms are negligibly small, and Eq. (23) generalizes Eq. (19) by more complicated dispersion terms.

TABLE II. Third and fourth order elastic constants in 10^{10} N/m^2 for some cubic crystals (after [36,37]) and contribution of the quadratic and cubic nonlinearities.

Material	c_{11}	c_{111}	c_{1111}	C_1/E^*	C_2/E^*
MgO	29.7	−489.5	7100	−6.7	23.9
RbCN	9.8	−149.1	1304	−6.1	7.46
LiCN	15.2	−190.1	−6785	−4.75	−86.4
NaCl	5	−86.5	759	−7.2	8.5
KCl	3.87	−71.3	1141	−7.7	31.2
V_3Si	29	-5.1×10^3	7.1×10^6	−85.6	4.04×10^4

B. Phenomenological model for strain waves in crystals

The coefficients in Eq. (18) for cubic crystals are obtained using Eq. (3) of the form

$$E^* = c_{11}, C_1 = \frac{1}{2}(3c_{11} + c_{111}),$$

$$C_2 = \frac{1}{6}(3c_{11} + 6c_{111} + c_{1111}).$$

Contrary to the higher-order Murnaghan moduli, the crystalline constants are known for many materials. Some of them are shown in Table II.

It follows from Table II that usually crystals exhibit normal nonlinear features, and contribution of cubic nonlinearity is less than that of the quadratic one. In particular, it happens for MgO used in experiments in Refs. [22,23], which justifies their model Eq. (4) with only quadratic nonlinearity being taken into account. However, there exists crystal V_3Si for which cubic nonlinearity should be used together with quadratic one. This crystal belongs to the A-15 structure compounds that call attention due to the interrelation of structural instability and high-temperature superconductivity. An unusual degree of anharmonicity for this crystal was observed in Ref. [38].

The abnormal crystalline nonlinear features allow us to suggest a generalization of the model equation Eq. (4) by

$$\rho_0 v_{tt} - c_{11} v_{xx} - \frac{1}{2}(3c_{11} + c_{111})(v^2)_{xx} - \frac{1}{6}(3c_{11} + 6c_{111} + c_{1111}) \times (v^3)_{xx} - 2\rho_0 c \gamma v_{xxxx} = 0. \tag{24}$$

One can note that similar equation has been obtained in a phenomenological way in Ref. [39] where a two atoms model was used to obtain the governing equation. However, the cubic nonlinear term coefficient in [39] is always lower in order than the quadratic term one and dispersion is of order of the cubic nonlinear term.

Recently an influence of magnetic field has been studied in Refs. [40,41] for paramagnetic crystals. The nonlinear and dispersion effects are governed by the intrinsic properties of the crystal and the spin-phonon interaction. As a result the governing equation for longitudinal strains was obtained there in the form of Eq. (19) with coefficients defined by

$$a = \frac{1}{\rho}\left(c_{11} - \frac{n_s G_{11}^2}{4\hbar\omega_0}\right); c_1 = \frac{1}{2\rho}(3c_{11} + c_{111}); c_2 = \frac{1}{6\rho}(3c_{11} + 6c_{111} + c_{1111}) + \frac{n_s G_{11}^4}{32\hbar^3\omega_0^3},$$

$$\alpha_3 = -\frac{n_s G_{11}^2}{16\hbar\omega_0^2\rho}, \quad \alpha_4 = \frac{h^2}{24\rho}\left(2c_{11} + \frac{5n_s G_{11}^2}{\hbar\omega_0}\right),$$

where h is a distance between neighboring atoms in a lattice, n_s is concentration of the impure paramagnetic ions, G_{11} is a constant of spin-phonon interaction, $\omega_0 = 2g\nu_B B_0/\hbar$, g is the Lande factor, ν_B is the Bohr magneton, B_0 is the external magnetic field directed perpendicular to the direction of the strain wave propagation, $\hbar$ is the Plank's constant.

In the absence of magnetic field the contribution of the cubic nonlinearity in the model Eq. (19) is negligibly small. In the presence of paramagnetic ions in the crystal the estimations were obtained in [40,41] for the crystal MgO with paramagnetic ions Fe^{2+} and Ni^{2+}. It turns out that the value of c_2 may be 4 orders higher than that of c_1. This is just the case realized for seismic materials mentioned above when both quadratic and cubic nonlinearities should be taken into account for description of longitudinal strain waves.

C. Models for media with nonlinear internal structure

Among essentially nonlinear models for the strains we first note the model for elastic ferroelectrics developed in Refs. [3,11,12]. A continuum model was obtained in the 3D case based on the assumption of a deformed chain of atoms modified by a rotating microstructure. Both the nonlinearity and dispersion are caused by a dipole rotation, while elastic macrodeformations were assumed small enough to consider them linearly elastic. When only longitudinal displacement $U(x,t)$ and rotation $\phi(x,t)$ in the plane perpendicular to the direction of the longitudinal wave propagation are taken into account, the density of energy may be written in the notations of Refs. [3,11,12] as

$$\Pi = 1/2(c_L^2 U_X^2 + \phi_X^2) + (\chi - \alpha_L U_X)[1 + \cos(\phi)].$$

Then the Hamilton variational principle gives rise to the coupled equations:

$$\rho U_{TT} - c_L^2 U_{XX} = \alpha_L [1 + \cos(\phi)]_X, \tag{25}$$

$$\phi_{TT} - \phi_{XX} = (\alpha_L U_X + \chi)\sin(\phi). \tag{26}$$

For small angles of rotation Eqs. (25) and (26) are transformed to Eqs. (15) and (16). Indeed, trigonometric functions are expanded in Taylor series, and only first terms in the expansions are left in Eqs. (25) and (26), so $\cos(\phi)$ is replaced by $1-\phi^2/2$, while the sin function is replaced with ϕ.

Less known is the model developed in Refs. [13,14] that considers a complex lattice of a crystal consisting of two sublattices generalizing the linear analog developed by Born and Huang [42]. Besides interatomic forces between atoms, the relative sublattices motion is taken into account in the model, hence it generalizes the well-known Frenkel-Kontorova model for the simple lattice to describe structural deviations in the biatomic lattice.

According to [13,14] the governing equations are obtained using a continuum approach without making a continuum limit of a discrete model similar to Refs. [3,12]. The equations are derived for the vectors of macrodisplacement U and relative microdisplacement u for the pair of atoms with masses m_1 and m_2,

$$\mathbf{U}=\frac{m_1\mathbf{U}_1+m_2\mathbf{U}_2}{m_1+m_2},\quad \mathbf{u}=\frac{\mathbf{U}_1-\mathbf{U}_2}{a},$$

where a is a period of sublattice. In general, it allows us to describe both translational and rotational motions of the internal structure. In the one-dimensional case, only translational motion is considered, and the Hamilton principle is employed with kinetic energy density K,

$$K=\frac{\rho U_t^2+\mu u_t^2}{2},$$

and the internal density energy Π,

$$\Pi=\frac{EU_x^2+\kappa u_x^2}{2}+(p-SU_x)[1-\cos(u)]. \tag{27}$$

Comparing it with the Murnaghan model (1), one can see the absence of the terms describing physical nonlinearity at the macro level. Instead, the last term in Eq. (27) accounts for the sublattices interaction. In the absence of coupling at $S=0$, the Frenkel-Kontorova model is revealed. Trigonometric functions allow us to describe an identity of the complex lattice after displacement proportional to the period of the sublattice.

Then the following coupled equations are obtained:

$$\rho U_{tt}-EU_{xx}=S[\cos(u)-1]_x, \tag{28}$$

$$\mu u_{tt}-\kappa u_{xx}=(SU_x-p)\sin(u). \tag{29}$$

Contrary to the phenomenological model developed in the previous subsection, the coefficients of both structural models are unknown. Some estimations may be given based on the fact that for wave processes the right-hand side in Eqs. (28) and (29) should be of lower order in comparison with those in the left-hand side. The function u may be of order 1, the macrostrain $v=U_x$ is of order 10^{-4}–10^{-5}, and the Young modulus is of order 10^{10} for most materials. Therefore, the value of the parameter S should be of order 10^6 as follows from Eq. (28) while $p\sim10^3$ follows from Eq. (29). Similar estimations may be done for the coefficients in Eqs. (25) and (26).

Both Eqs. (25) and (26) and Eqs. (28) and (29) may be decoupled and transformed to a single governing equation in two ways. The authors of the models [3,11–14] obtained the double Sine-Gordon equation for a variable characterizing the *micromotion*. The ODE reduction of Eqs. (28) and (29) for the traveling wave solutions depending only on the phase variable, $\theta=x-Vt$ is obtained from Eqs. (28) and (29) after some algebraic manipulations,

$$u_\theta^2=g+2p_1^*(1-\cos u)-p_2^*(1-\cos u)^2, \tag{30}$$

where g is a constant of integration and

$$p_1^*=\frac{p}{\mu(c_l^2-V^2)}\left(1-\frac{\sigma S}{p\rho(c_L^2-V^2)}\right),$$

$$p_2^*=\frac{S^2}{\mu\rho(c_l^2-V^2)(c_L^2-V^2)}, \tag{31}$$

where $c_L^2=E/\rho$, $c_l^2=\kappa/\mu$, σ is another constant of integration. The solitary wave solutions correspond to the choice $u=0$ at infinities giving $g=0$. The choice $u=\pm\pi$ at infinity corresponds either to the solitary wave or a kink of micromotion, in this case $g=4(p_2^*-p_1^*)$.

Now the governing equation for a *macrostrain* is obtained. In this case Eq. (28) is resolved for $\cos(u)$,

$$\cos(u)=1-\frac{(E-\rho V^2)U_x-\sigma}{S}, \tag{32}$$

where σ is the same as in Eqs. (31). Then it follows from Eq. (29) that

$$v_\theta^2=a_0+a_1v+a_2v^2+a_3v^3+a_4v^4, \tag{33}$$

which is the same as Eq. (20) derived in the framework of a phenomenological approach. Now all coefficients a_i depend on the velocity V,

$$a_0=\frac{\sigma^2(2S+\sigma)\kappa[S\sigma-2p\rho(c_L^2-V^2)]}{\mu\rho S^3(V^2-c_l^2)(c_L^2-V^2)}-\frac{\sigma\kappa g(2S+\sigma)}{S^2},$$

$$a_1=\frac{2p\sigma(2S+\sigma)}{S\mu\rho(c_L^2-V^2)(V^2-c_l^2)}-\frac{2a_0S^2(S+\sigma)}{\kappa\rho(c_L^2-V^2)\sigma\,(2S+\sigma)]},$$

$$a_2=\frac{4p\rho(c_L^2-V^2)(S+\sigma)+S\sigma(2S+\sigma)}{\mu\rho S(c_L^2-V^2)(c_l^2-V^2)}+\frac{a_0S^2}{\kappa\sigma(2S+\sigma)},$$

$$a_3=\frac{2[p\rho(c_L^2-V^2)+S(S+\sigma)]}{S\mu(V^2-c_l^2)},\quad a_4=\frac{\rho(c_L^2-V^2)}{\mu(c_l^2-V^2)}.$$

The term v_θ^2 reflects dispersion that is caused by coupling like nonlinearity. The estimations for the coefficients done before allow us to check whether contribution of quadratic and cubic nonlinear term is of the same order. Indeed, the nonlinear term coefficients ratio $a_3/a_4=\max\, p/S, S/E$, turns out of order 10^{-3}, which yields almost equal contribution of

TABLE III. Signs of the wave parameters and types of the waves for $\sigma=0$.

V^2	$(0;c_L^2-c_0^2)$	$(c_L^2-c_0^2;c_L^2)$	$(c_L^2;c_L^2+c_0^2)$	$>c_L^2+c_0^2$
A	>0	>0	>0	<0
Q_+	>0	$(-1;0)$	<-1	>0
Q_-		>0		
Wave	Tensile (34)	Tensile (34)	Compression (35)	Compression (34)

quadratic and cubic nonlinearities for typical elastic strains of order 10^{-4}. Equation (33) possesses solutions vanishing at infinity, $|\theta|\to\infty$, together with its derivatives provided that $a_0=0$ and $a_1=0$. The former happens for $\sigma=0$ or $\sigma=-2S$ while the latter requires also $a_0/[\sigma(2S+\sigma)]=0$. The last condition is realized provided that $g=0$ for $\sigma=0$ and $g=4(p_2^*-p_1^*)$ for $\sigma=-2S$ that coincides with the analysis done above for the solutions of the Eq. (30).

IV. TRAVELING WAVE SOLUTIONS

A reasonable question arises after establishing a similarity in the governing equations for the macrostrains: is there a need in the study of a more complicated structural model with unknown coefficients rather than a simpler phenomenological model whose parameters it is possible to measure? An analysis of exact localized traveling wave solutions will be used in this section to answer it.

A. Localized strain waves for the phenomenological model

When $\alpha_0=0$, $\alpha_1=0$, ODE (20) for the phenomenological model possesses exact solutions of two kinds that may be obtained by direct integration [35,43,44]

$$v_1=\frac{A}{Q\cosh(k\theta)+1}, \tag{34}$$

$$v_2=-\frac{A}{Q\cosh(k\theta)-1}, \tag{35}$$

where

$$A=\frac{3(V^2-a)}{c_1},\quad Q=\sqrt{1+\frac{9c_2}{2c_1^2}(V^2-a)},\quad k^2=\frac{V^2-a}{\alpha_4-\alpha_3V^2}. \tag{36}$$

The solution v_2 is bounded for positive values of c_2, which happens for all seismic media studied in Refs. [7–9,31]. The amplitude of the solutions is always of either sign that makes possible the *simultaneous* existence of compression and tensile strain waves. This is the most important difference from the weakly nonlinear case considered in Sec. I.

These waves are generated and interact in different ways as discovered by numerical simulations in Refs. [45–48]. Exact solutions (34) and (35) need a specific initial condition as their forms at $t=0$. However, for $c_1>0$, a rather arbitrary initial pulse splits into a sequence of tensile solitary waves each described by the exact solution (34) [45]. It is found that less time is required for solitary wave formation at positive value of c_2 than that of the DDE solitary wave formation. The value of c_2 affects the amplitude and the velocity of the wave but not the number of solitary waves arising from the input. Thus we found that the larger is the positive value of c_2 the higher and faster are the arising solitary waves. The number of generated solitary waves depends upon the mass of the input. Cubic nonlinearity does not affect the number of solitary waves.

On the contrary, with decreasing negative value of c_2 we obtain that smaller solitary waves propagate more slowly, and more time is needed for the formation of the solitary waves. A threshold value of negative c_2 is found after which a breaking down of the initial wave happens, and no solitary wave appears. This value is equal to the limiting value $-2c_1/3c_2$ found from the analysis of the exact solution. Our results confirm the prediction done on the basis of exact solutions that no tensile solitary waves exist for $c_2<0$ and $c_1<0$, while no compression waves exists for $c_2<0$ and $c_1>0$ [remember that Eq. (35) does not describe bounded solutions in this case]. The oscillating wave packets arise instead of the trains of localized strain waves.

Another evolution is observed for $c_2>0$, when the formation of the waves (35) turns out to be possible [46]. In particular, a generation of tensile waves happens for $c_1<0$, and of compression waves for $c_1>0$. No generation of the localized waves happens for small values of c_2; here an evolution happens according to solution (34). However, a tensile solitary wave arises above some threshold value of c_2. Contrary to the formation of the waves (34), the value of c_2 affects the number of generated solitary waves (35). Increase in an initial velocity of the input results in an increase in the number of generated waves. Like for the formation of the waves (34), the number of generated waves (35) depends upon the value of c_1.

Interaction of strain solitary waves is studied in Refs. [47,48]. It was found that the waves of the same kind interact like the solitary waves of the KdV equation for a takeover interaction, and like the solitary waves of the Boussinesq equation at the head-on interaction. However, waves of different kind demonstrate another evolution on the takeover interaction that already similar to that of the Gardner equation [44].

When the last nonlinear terms in Eq. (23) are negligibly small, its traveling wave solution is obtained from the ODE similar to Eq. (20). Then the parameters of solutions (34) and (35) are defined by

TABLE IV. Signs of the wave parameters and types of the waves for $\sigma=-2S$.

V^2	$(0;c_L^2-c_0^2)$	$(c_L^2-c_0^2;c_L^2)$	$(c_L^2;c_L^2+c_0^2)$	$>c_L^2+c_0^2$
A	<0	>0	>0	>0
Q_+	>0	<-1	$(-1;0)$	>0
Q_-			>0	
Wave	Compression (34)	Compression (35)	Tensile (34)	Tensile (34)

$$A=\frac{18[B(\rho V^2-A)+D^2]}{BN},$$

$$Q=\sqrt{1+\frac{27M\,[B(\rho V^2-A)+D^2]\}}{2BN^2}}. \qquad (37)$$

Again the amplitudes of the solutions are of either sign. Solution (35) is bounded for positive values of M provided that $B(\rho V^2-A)+D^2>0$.

B. Localized strain waves for the structural models

At first glance, the solutions of Eq. (33) at $a_0=0$, $a_1=0$ possess the same properties as those of a phenomenological model. Indeed, exact solutions of two types (34) and (35) may be obtained, where for $\sigma=0$ with

$$A=\frac{4S}{\rho(c_0^2+c_L^2-V^2)},\quad Q_\pm=\pm\frac{c_L^2-V^2-c_0^2}{c_L^2-V^2+c_0^2},$$

$$k=2\sqrt{\frac{p}{\mu(c_l^2-V^2)}} \qquad (38)$$

and for $\sigma=-2S$

$$A=\frac{4S}{\rho(c_0^2+V^2-c_L^2)},\quad Q_\pm=\pm\frac{V^2-c_L^2-c_0^2}{V^2-c_L^2+c_0^2},$$

$$k=2\sqrt{\frac{p}{\mu(V^2-c_l^2)}}, \qquad (39)$$

where $c_0^2=S^2/(p\rho)$.

However, the dependence of the equation coefficients on velocity gives rise to different predictions about the existence and the kind (tensile or compression) of solutions. First, for $\sigma=0$ we have $V^2<c_l^2$, while for $\sigma=-2S-V^2>c_l^2$. In both cases the solution of the first kind (34) is realized for $Q_\pm>0$, while *bounded* solution of the second kind (35) appears at $Q_\pm<-1$.

The sets of the parameters (38) and (39) correspond to physically different processes that follow from the relationship between micro- and macrostrains, Eq. (32). A substitution of the solutions with parameters defined by Eq. (38) into Eq. (32) demonstrates that for $\sigma=0$, $u=0$ at $|\theta|\to\infty$, which means the absence of a microstructure in the absence of the macrostrain wave. For $Q=Q_+$ at the point of maximum/minimum, $\theta=0$, there is a shift of sublattice that is less than or equal to the half of its period, $u(0)\le\pi$. For $Q=Q_-$ such a shift is realized provided that

$$c_L^2-c_0^2\le V^2\ge c_L^2, \qquad (40)$$

Similarly, $u=\pi$ at $|\theta|\to\infty$ for $\sigma=-2S$ that corresponds to a preliminary shift by a half-period before macrostrain solitary waves comes. Passing the point of maximum/minimum, $\theta=0$, yields an additional shift by a half-period of the sublattice for $Q=Q_+$, while for $Q=Q_-$ a shift is realized when the velocity of the wave lies within the interval

$$c_L^2\le V^2\le c_L^2+c_0^2. \qquad (41)$$

Therefore, no simultaneous existence of the localized macrowaves, corresponding to the choices $\sigma=0$ and $\sigma=-2S$, is possible since different boundary conditions for macrodisplacements at infinity, $u=0$ or $u=\pi$, are needed at one time.

For each value of σ an analysis of solutions (34) and (35) gives rise to the dependence of the possible kind of localized strain macrowave on its velocity. The results for $\sigma=0$ and $\sigma=-2S$ are summarized in Tables III and IV correspondingly. The absence of waves with Q_- for some intervals is caused by the restrictions of the boundary conditions (40) and (41). Certainly, permitted intervals for the velocity should be checked, $V^2<c_l^2$ for $\sigma=0$, and $V^2>c_l^2$ for $\sigma=-2S$. The most important conclusion following from these tables is the absence of simultaneous existence of the compression and tensile waves for both values of σ that differs from the case of essentially nonlinear waves studied in the previous section in a phenomenological way.

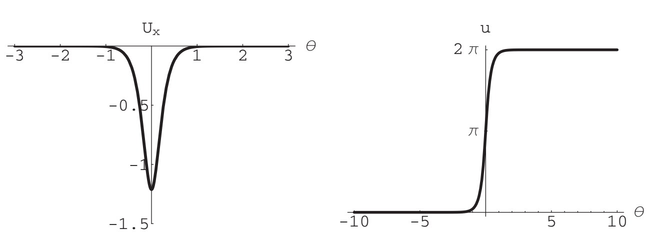

FIG. 1. Compression macrostrain wave and corresponding wave of microstrain in the interval $c_L^2+c_0^2<V^2<c_l^2$.

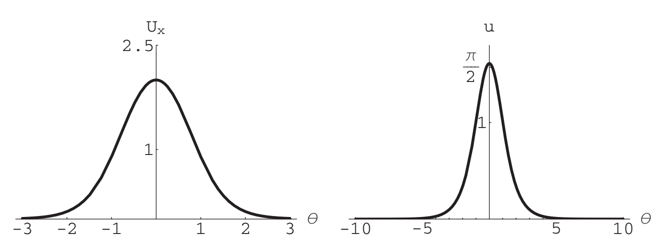

FIG. 2. Compression macrostrain wave and corresponding wave of microstrain in the interval $c_L^2-c_0^2<V^2<c_L^2$, $V^2<c_l^2$.

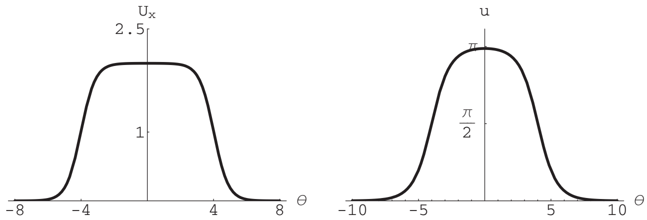

FIG. 3. “Fat” compression macrostrain wave and corresponding wave of microstrain when $V^2 \to c_L^2 - c_0^2$.

The sign of the amplitude of the wave $A/(Q_\pm+1)$ depends on the relationship between V, c_L, and c_0. However, the variation in the value of the amplitude of the wave $A/(Q_-+1)$ does not depend on the variation in the velocity V while $A/(Q_++1)$ does. These variations are different for different values of σ. Thus, for $\sigma=0$ the amplitude of the tensile wave $A/(Q_\pm+1)$ increases with increase in the velocity, while that of the compression wave decreases. For $\sigma=-2S$ all is inverted. A special case corresponds to the so-called “fat” wave that appears as $Q_\pm$ tends to zero. In this case the width of the wave increases without limit while its amplitude tends to the finite value equal to A. It is called a “fat” solitary wave. For $\sigma=0$ this specific case is realized when the velocity tends to $\sqrt{c_L^2-c_0^2}$, while for $\sigma=-2S$ we obtain the “fat” wave solution when the velocity tends to $\sqrt{c_L^2+c_0^2}$.

An analysis of the shapes of the exact solutions also should take into account the coupling governed by Eq. (32). Depending upon the value of the first derivative at $\theta=0$, one can express it in a different way. Reversing the cos function for derivation of the expression for u, one has to avoid the point where the first derivative does not exist. This breaking happens for $\theta=0$ at $\sigma=0$ and for $Q=Q_+$. Therefore, the solution for u obtained using both Eqs. (34) and (35) should be written as

$$u = \arccos\left(\frac{(\rho V^2 - E)U_x}{S} + 1\right) \quad \text{for} \quad \theta \leq 0, \tag{42}$$

$$u = 2\pi - \arccos\left(\frac{(\rho V^2 - E)U_x}{S} + 1\right) \quad \text{for} \quad \theta > 0. \tag{43}$$

Similar conclusions may be done using the phase diagrams analysis of Eq. (30), e.g., like in Ref. [11].

Typical profiles for the waves with velocities from the interval $c_L^2+c_0^2 < V^2 < c_l^2$ are shown in Fig. 1. Similar profiles arise for $c_L^2 < V^2 < c_L^2 + c_0^2, V^2 < c_l^2$, however, a singularity in the solution appears when $V \to c_L$.

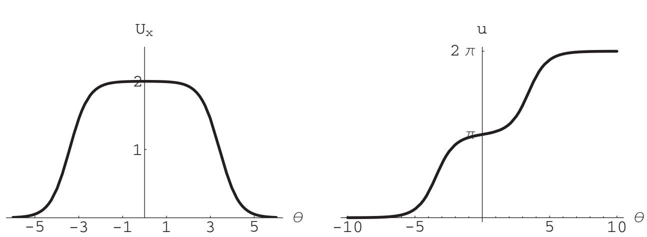

FIG. 4. “Fat” macrostrain compression wave and corresponding microstrain wave for $V^2 \to c_L^2 - c_0^2$.

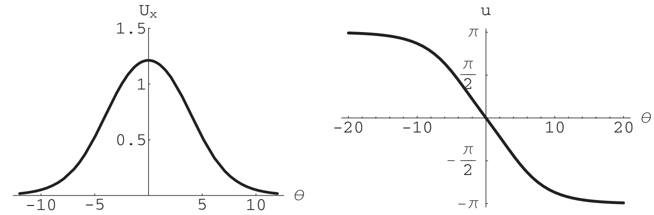

FIG. 5. Tensile macrostrain wave and corresponding microstrain wave with the velocities from the interval $c_L^2+c_0^2 < V^2$, $V^2 > c_l^2$.

The profiles corresponding to the interval $c_L^2-c_0^2 < V^2 < c_L^2, V^2 < c_l^2$ are shown in Fig. 2. Now the first derivative is zero for $Q=Q_-$ at $\theta=0$, and the expression for u reads

$$u = \arccos\left(\frac{(\rho V^2 - E)U_x}{S} + 1\right). \tag{44}$$

One can see in Fig. 2 a changing of the kind of the macrostrain wave to the tensile wave. The profile of the microstrain now describes local perturbation of the internal structure with coming back to the initial undisturbed state.

A “fat” solitary wave of macrostrain is shown in Fig. 3. Note that the amplitude of the macrostrain wave is indifferent to the tendency of the velocity to the threshold of the “fat” wave regime. On the contrary, a microstrain “fat” wave growths by the value corresponding to the half-period of the sublattice. Another shape of the “fat” wave is realized when $V^2 \to c_L^2 - c_0^2$ from below. One can see in Fig. 4 that the profile for the macrostrain “fat” wave is similar to that of the wave shown in Fig. 3. However, the corresponding microstrain wave has not a bell-shaped form anymore, now it describes the shift by the period of the sublattice. However, its shape differs from that one shown in Fig. 1.

For $\sigma=-2S$ and $Q=Q_+$ the first derivative u_θ is not zero at $\theta=0$, and the solution for u corresponding to solution (34) reads

$$u = \arccos\left(\frac{(\rho V^2 - E)U_x}{S} - 1\right) \quad \theta \leq 0, \tag{45}$$

$$u = -\arccos\left(\frac{(\rho V^2 - E)U_x}{S} - 1\right) \quad \theta > 0. \tag{46}$$

Typical profile of the wave is shown in Fig. 5, where the profile of the microstrain wave describes transformation from one nanostructure to another. Again the regime of the “fat” wave is possible as shown in Fig. 6 where the profile of

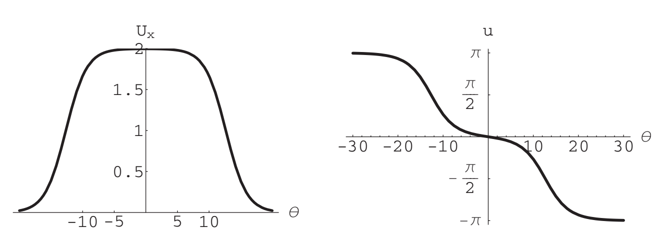

FIG. 6. “Fat” tensile macrostrain wave and corresponding microstrain wave with V^2 tends to $c_L^2+c_0^2$ from above.

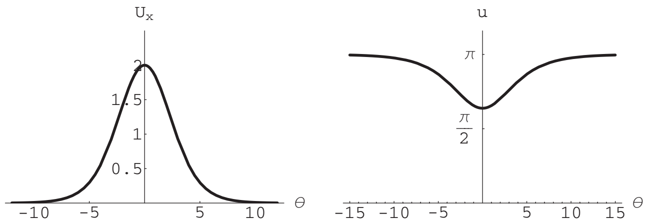

FIG. 7. Tensile macrostrain wave and corresponding microstrain wave in the interval $c_L^2 < V^2 < c_L^2 + c_0^2$, $V^2 > c_l^2$.

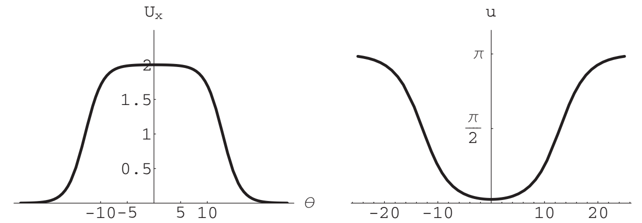

FIG. 8. "Fat" tensile macrostrain wave and corresponding microstrain wave with V^2 tends $c_L^2 + c_0^2$ from below.

the microstrain wave has a "stepwise" shape. For $\sigma = -2S$ and $Q = Q_-$ we have for u

$$u = \arccos\left(\frac{(\rho V^2 - E)U_x}{S} - 1\right). \tag{47}$$

Then a wave of macrostrain gives rise to the perturbation of the shift of sublattice in the interval $c_L^2 < V^2 < c_L^2 + c_0^2, V^2 > c_l^2$, see Fig. 7, while Fig. 8 demonstrates the "fat" macrostrain wave generating a corresponding microstrain wave. Again the amplitude of the macrostrain wave does not depend on velocity, while the microstrain wave tends to the undisturbed value for $\theta = 0$ as the velocity tends to the threshold value for the "fat" wave.

C. Dispersive features of phenomenological and structural models

Dispersive properties of the phenomenological model Eq. (19) follows from the last expression of Eq. (36),

$$V^2 = \frac{a + \alpha_4 k^2}{1 + \alpha_3 k^2}.$$

Here the mixed dispersion provides a threshold value for the velocity equal to $\sqrt{\alpha_4/\alpha_3}$ as $k \to \infty$. Certainly the wave belongs to the acoustic mode. The solution of the generalized model Eq. (23) gives rise to a more complicated dispersion relation following from the last expression in Eq. (37),

$$k^2 V^4 + \left[\frac{B}{I} - k^2\left(\frac{C}{I} + \frac{A}{\rho}\right)\right] V^2 + \frac{AC}{\rho I} k^2 + \frac{D^2}{\rho I} - \frac{AB}{\rho I} = 0.$$

It possesses two kinds of solutions; see [19] for details. The most important is that one of them belongs to the optic mode while another one belongs to the acoustic mode.

Linearized structural models [Eqs. (25) and (26) and Eqs. (28) and (29)] do not exhibit dispersion for the macrostrains. There exists generalization of the model [Eqs. (25) and (26)] as well as of its weakly nonlinear analog [Eqs. (15) and (16)] taking into account shear macrostrains. In this case dispersion relation contains both acoustic and optic branches while longitudinal strain waves remain nondispersive.

The last expressions in Eqs. (38) and (39) demonstrate dispersion for the macrostrain wave,

$$V^2 = c_l^2 - \frac{4p}{\mu k^2},$$

$$V^2 = c_l^2 + \frac{4p}{\mu k^2},$$

for $\sigma = 0$ and $\sigma = -2S$, respectively. Therefore, in the presence of nonlinear coupling, macrostrain wave acquires dispersion of the microstrain optical mode.

V. CONCLUSIONS

To sum up, two approaches for the materials with internal structure are considered and compared. In the first, both dispersion and essential nonlinearity are described in a phenomenological way. It is developed for the seismic materials and paramagnetic materials but it might be formally extended for the materials with micro- or even nanostructure. Another approach is developed for the crystals with deviations at the nanolevel. According to it, both dispersion and essential nonlinearity for the macrostrain wave may be caused by variations in the internal structure of a crystal.

An important finding is that the governing equation for traveling macrostrain waves is the almost the same for both approaches. It points to an advantage of the phenomenological approach since it is simpler, and the parameters of the model can be measured. However, a comparison of the exact solutions reveals important deviations that demonstrate a need in the theory taking into account the proper internal structure of the material. They concern the absence of simultaneous existence of the waves allowed by the phenomenological theory, as well as strong dependence of the kind of the wave on the value of its velocity. The localized waves possess optic dispersion features while phenomenological theory predicts existence of the macrostrain wave whose dispersion belongs to the acoustic branch.

More important is that the proper structural theories allow us to reveal the behavior of the microstrains that describe structural deviations in the crystalline lattice, formation of defects, etc. These structural deviations might be predicted by observation of the evolution of macrostrain solitary waves. Future studies focus on the numerical study of the formation of the waves in the framework of the structural models as well on the search of the possibilities to measure the value for the parameters of the models.

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